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SUBCRITICALITY AND GAUGEABILITY OF THE SCHRÖDINGER OPERATOR

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ABSTRACT. We investigate a Schrödinger operator $-\Delta/2 + V$ in R^d $(d \geq 3)$ with a potential V in the class K_d satisfying a similar Kato condition at infinity, and prove an equivalence theorem connecting various conditions on subcriticality, strong positivity and gaugeability of the operator.

1. INTRODUCTION

We consider the Schrödinger operator $H = -\Delta/2 + V$ in R^d $(d \geq 3)$ from three different aspects: criticality or subcriticality, strong positivity, and gaugeability. In each aspect, we present several conditions on V (or H). The purpose of this paper is to prove that all these conditions in the three aspects are equivalent for a large class of potentials V.

We first assume that the potential V is in Kato class K_d, i.e. V is a Borel measurable function in R^d satisfying

$$(1) \qquad \lim_{r\downarrow 0}\left[\sup_{x\in R^d}\int_{|y-x|\leq r}\frac{|V(y)|}{|y-x|^{d-2}}\,dy\right] = 0,$$

or more generally, V is in K_d^{loc} i.e., for any ball B, $1_B V \in K_d$, where 1_B denotes the indicator function of B.

In §2, we define a new subclass K_d^∞ of K_d and show that K_d^∞ contains $K_d \cap L^1(R^d)$ and any potentials behaving like $c|x|^{-\rho}$ $(\rho > 2)$ near infinity. In §4, we shall prove the main equivalence theorem for any potential V in K_d^∞ after presenting a sequence of preliminary lemmas and propositions in §3.

In the rest of this section we describe these conditions in the three aspects respectively and discuss the key ideas which connect them.

(i) *Criticality or subcriticality*: There have been various notions for criticality. One of them, perhaps the most intuitive one, is that $-\Delta/2 + V \geq 0$ but for any essentially nonzero function $q \leq 0$, $-\Delta/2 + V + q \not\geq 0$. Under the assumption $H \geq 0$, if any notion for criticality is proposed then the opposite condition will be for subcriticality. Therefore in the following we only consider the subcritical condition. For instance, subcriticality corresponds to the idea described above

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is that the operator $H \geq 0$ allows a "small" perturbation and still keeps its nonnegativity. The precise notions for this are conditions (a) and (a1)–(a3) in MT, here and below, MT stands for the main theorem in §4.

There have been two other notions of subcriticality. (Simon [S1]) there exists a $\beta > 0$ such that $-\Delta/2 + (1+\beta)V \geq 0$ ((b) in MT); and (Murata [M2]) there exists a positive Green function $G^V(\cdot, \cdot)$ for H ((c) in MT).

Some implications and equivalences among these subcritical conditions have been proved under different assumptions on V. See [A1, M1, Pi]. A Green function approach for the Schrödinger operators has been given in [H-Z].

(ii) *Strong positivity*: By the Allegretto-Piepenbrink theory (see [S2]) for $V \in K_d^{\text{loc}}$, $H = -\Delta/2 + V \geq 0$ if and only if there exists a positive solution $u > 0$ of $Hu = 0$ in R^d. In view of this, one may expect to use the existence of some strongly positive solution u to characterize the subcriticality, for instance, the existence of a solution $u \geq c > 0$ of $Hu = 0$ (see (d), (d1), and (d2) in MT).

(iii) *Gaugeability*: It is known that the probabilistic counterpart of the Schrödinger operator $-\Delta/2 + V$ is the Feynman-Kac path integral. In this paper we consider the stopped Feynman-Kac integral in the following form:

$$\mathbb{E}^x \left[\exp\left(-\int_0^\xi V(Xs)\, ds \right) \right],$$

where $\{X_s\}$ denotes a Markov process in R^d with its lifetime ξ and $\mathbb{E}^x$ denotes the expectation over the paths starting with $x \in R^d$. We take the Markov process $\{X_s\}$ as the standard Brownian motion in R^d and the y-conditioned Brownian motion for a point y in R^d (i.e., $G(y, \cdot)$-process of Doob type see [D]), respectively and use the expectation E^x and E^x_y to distinguish them. We define the gauge and the conditional gauge respectively by

$$u_0(x) = E^x \left[\exp\left(-\int_0^\infty V(Xs)\, ds \right) \right], \tag{2}$$

and

$$u_0(x, y) = E^x_y \left[\exp\left(-\int_0^\xi V(Xs)\, ds \right) \right]. \tag{3}$$

The gaugeability means the finiteness or boundedness of these quantities (see (e), (e1), (f), and (f1) in MT). In recent years there has been an intensive study on the equivalence of finiteness and boundedness of the gauge and the conditional gauge mainly for a bounded domain in R^d (see [Ch-R, C-F-Z, F, Z–Z3]). In this paper we prove these equivalence for the whole space R^d and a potential $V \in K_d^\infty$ in a similar manner. This part is a continuation of the previous study.

The essential part of the main result lies in the connection between the three aspects. An equality given in [Z3]

$$\frac{G^V(x, y)}{G(x, y)} = E^x_y \left[\exp\left(-\int_0^\xi V(Xs)\, ds \right) \right],$$

reveals a connection between the existence of a positive $G^V(\cdot, \cdot)$ and the finiteness of the conditional gauge. But we need a key assertion as a critical linkage among these conditions.

For $V \in K_d^\infty$ and a large number $A > 0$, we define a shuttle time σ for any Brownian path starting at a point on $\partial B(0, A)$ as the first time returning to $\partial B(0, A)$ after hitting $\partial B(0, 2A)$, and a shuttle operator S_V in the Banach space $C(\partial B(0, A))$:

$$S_V f(x) = E^x \left[\sigma < \infty ; \exp\left(-\int_0^\sigma V(Xs)\,ds\right) f(X(\sigma))\right]. \tag{4}$$

Put

$$\lambda(V) = \lim_{n\to\infty} \sqrt[n]{\|(S_V)^n\|}, \tag{5}$$

which is a higher dimensional analogy of the quantity $u(a, b)u(b, a)$ given in [Ch] and [Ch-V].

Introducing the shuttle operator and its spectral radius $\lambda(V)$ is the key idea in connecting the seemingly different conditions. In fact we add a new equivalent condition on V into the list as a central linkage: $\lambda(V) < 1$, (see (g) in MT).

For one dimensional case $(d = 1)$, in a joint paper with F. Gesztesy [G-Z], we proved the equivalence theorem of subcriticality for the general Sturm-Liouville operators, which becomes an important motivation of this paper. For $d = 2$, if we consider a transient domain like $B_r^* = \{x \in R^2 : |x| > r\}$ $(r > 0)$, then we can extend the main result to this transient domain, but if we also consider the whole space R^2, then there is an essential difference with the case $d \geq 3$ due to the recurrence of R^2. A conjecture may be proposed: $-\Delta/2 + tV$ in R^2 is critical for each $t \in [0, 1]$ (or only for $t = t_1$ and t_2, $0 < t_1 < t_2$) if and only if $V \equiv 0$ a.e. in R^2. The same statement in the one dimensional case has been proved in [G-Z].

2. Class K_d^∞ and basic notions

We assume that the dimension $d \geq 3$. For any domain $D \subseteq R^d$, $L^p(D)$ $(1 \leq p \leq \infty)$ are relative to the Lebesgue measure and put

$$\mathscr{B}(D) = \{\text{all Borel measurable functions in } D\}.$$

For any compact set Λ in R^d, $C(\Lambda)$ denotes the Banach space of all continuous functions on Λ with the maximum norm.

The basic classes of potentials are given by

$$K_d = \left\{ V \in \mathscr{B}(R^d) : \lim_{a\downarrow 0} \left[\sup_{x\in R^d} \int_{|y-x|\leq a} \frac{|V(y)|}{|y-x|^{d-2}}\,dy \right] = 0 \right\}, \tag{6}$$

and

$$K_d^{\text{loc}} = \{ V \in \mathscr{B}(R^d) : \forall \rho > 0,\ 1_{B(0,\rho)} V \in K_d \}, \tag{7}$$

(see [A-S] or [S2]).

We now introduce a new function class

Definition 1.

$$K_d^\infty = \left\{ V \in K_d^{\text{loc}} : \lim_{A\uparrow\infty} \left[\sup_{x\in R^d} \int_{|y|\geq A} \frac{|V(y)|}{|y-x|^{d-2}}\,dy \right] = 0 \right\}. \tag{8}$$

Proposition 1. $K_d \cap L^1(R^d) \subseteq K_d^\infty \subseteq K_d$.

Proof. Suppose $V \in K_d \cap L^1(R^d)$. For any $\varepsilon > 0$, there is an $a > 0$ such that

$$\sup_{x\in R^d} \int_{|y-x|\le a} \frac{|V(y)|}{|y-x|^{d-2}}\,dy < \frac{\varepsilon}{2}.$$

Since $V \in L^1(R^d)$, there is an $A > 0$ such that

$$\int_{|y|\ge A} |V(y)|\,dy < a^{d-2}\frac{\varepsilon}{2}.$$

Then we have for each $x \in R^d$,

$$\begin{aligned}\int_{|y|\ge A} \frac{|V(y)|}{|y-x|^{d-2}}\,dy &\le \int_{|y-x|\le a} \frac{V(y)}{|y-x|^{d-2}}\,dy + \frac{1}{a^{d-2}}\int_{|y|\ge A} |V(y)|\,dy \\ &\le \frac{\varepsilon}{2} + \frac{1}{a^{d-2}}\left(a^{d-2}\frac{\varepsilon}{2}\right) = \varepsilon.\end{aligned}$$

So $V \in K_d^\infty$.

The inclusion $K_d^\infty \subseteq K_d$ follows from

$$\begin{aligned}\{(x,y) : |y-x| \le a\} &\subseteq \{(x,y) : |y-x| \le a \text{ and } |x| \le A+1\} \\ &\quad \cup \{(x,y) : |y| \ge A\}, \quad \text{for } 0 < a \le 1.\end{aligned}$$ □

Proposition 2. *For* $\rho > 2$, $\{V \in K_d : V(x) = O(|x|^{-\rho}) \text{ as } |x| \to \infty\} \subseteq K_d^\infty$.

Proof. Since $\rho > 2$, we can choose a number $p > 1$ such that $d/2 > p > d/\rho$. This implies that $\rho p > d$ and $(d-2)p/(p-1) > d$.

For any $\varepsilon > 0$, since $V \in K_d$, $\exists a > 0$ such that

$$\sup_x \int_{|y-x|\le a} \frac{|V(y)|}{|y-x|^{d-2}}\,dy \le \varepsilon.$$

We suppose that $\exists c > 0$ such that for sufficiently large A, when $|y| \ge A$, $|V(y)| \le c/|y|^\rho$. Then by Hölder's inequality, for each $x \in R^d$,

$$\begin{aligned}\int_{|y|\ge A} \frac{|V(y)|}{|y-x|^{d-2}}\,dy &\le \varepsilon + c\int_{\substack{|y|\ge A \text{ and} \\ |y-x|\ge a}} \frac{dy}{|y-x|^{d-2}|y|^\rho} \\ &\le \varepsilon + c\left(\int_{|y|\ge A}\frac{dy}{|y|^{\rho p}}\right)^{1/p}\left(\int_{|y-x|\ge a}\frac{dy}{|y-x|^{(d-2)p/(p-1)}}\right)^{(p-1)/p} \\ &= \varepsilon + c\left(\int_{|y|\ge A}\frac{dy}{|y|^{\rho p}}\right)^{1/p}\left(\int_{|y|\ge a}\frac{dy}{|y|^{(d-2)p/(p-1)}}\right)^{(p-1)/p}.\end{aligned}$$

Therefore we obtain

$$\varlimsup_{A\uparrow\infty}\left[\sup_{x\in R^d}\int_{|y|\ge A}\frac{|V(y)|}{|y-x|^{d-2}}\,dy\right] \le \varepsilon.$$

So we conclude $V \in K_d^\infty$. □

For $V \in K_d^\infty$, put

$$|||\, V \,||| = \sup_{x\in R^d}\int_{R^d}\frac{|V(y)|}{|y-x|^{d-2}}\,dy. \tag{9}$$

Obviously, we have $|||\, V \,||| < \infty$. In fact, $(K_d^\infty, |||\cdot|||)$ forms a Banach space, but we do not need this point later.

It is easy to verify the following proposition by definition:

Proposition 3. *Let* $V \in \mathscr{B}(R^d)$. *Then we have*

(i) $V \in K_d^{\mathrm{loc}}$ *if and only if for any bounded domain* D *in* R^d, *the family* $\{|V(y)|/|x-y|^{d-2}, x \in D\}$ *is uniformly integrable in* D.

(ii) $V \in K_d^{\infty}$ *if and only if the family* $\{|V(y)|/|x-y|^{d-2}, x \in R^d\}$ *is uniformly integrable in* R^d.

Let $\{X_t : t \geq 0\}$ be the Brownian motion in R^d on the probability space $(\Omega, \mathscr{F}, P)$ and let P^x and E^x denote the probability and the expectation over the Brownian paths starting with $x \in R^d$, respectively. For any $A \in \mathscr{F}$ and a positive measurable function f on Ω, we put

$$E^x[A; f] = \int_A f(\omega) P^x(d\omega).$$

For $V \in K_d$, put

$$e_V(t) = \exp\left[-\int_0^t V(Xs)\,ds\right], \qquad 0 \leq t \leq \infty. \tag{10}$$

For any domain D in R^d $(d \geq 3)$, let $G_D(x, y)$ be the Green function in D corresponding to $-\Delta/2$. Then for each $y \in D$, we can define the conditional y-process in $D\backslash\{y\}$ by the transition density: (see [D, Z3])

$$p_{y,D}(t, x, w) = G_D(x, y)^{-1} p_D(t, x, w) G_D(w, y), \quad \text{for } t > 0,\ x, w \in D\backslash\{y\}, \tag{11}$$

where p_D is the transition density for the killed Brownian motion outside D (see [P-S]). We shall use $P^x_{y,D}$ and $E^x_{y,D}$ to denote the probability and the expectation over the y-conditioned paths in D starting with x, respectively. If $D = R^d$, then we shall omit D, so they become P^x_y and E^x_y.

We mainly consider the following subdomains of R^d: for $r > 0$,

$$B_r = B(0, r) = \{x \in R^d : |x| < r\},$$

and

$$B_r^* = \{x \in R^d : |x| > r\}.$$

For each $r > 0$, the Green function and the Poisson kernel of B_r and B_r^* corresponding to $-\Delta/2$ have the same form. Let D_r be either B_r or B_r^*. Then

$$G_{D_r}(x, y) = \frac{c_d}{|x-y|^{d-2}} - \frac{c_d r^{d-2}}{|y|^{d-2}|y^* - x|^{d-2}}, \qquad x, y \in D_r, \tag{12}$$

where $y \neq 0$, $y^* = r^2 y/|y|^2$ and $c_d = \Gamma(\frac{d}{2} - 1)/2\pi^{d/2}$;

$$K_{D_r}(x, z) = c_d(d-2)\frac{|r^2 - |x|^2|}{r|x-z|^d}, \qquad x \in D_r,\ z \in \partial D_r. \tag{13}$$

Remark. When $D_r = B_r^*$, the Poisson kernel $K_{D_r}(x, z)$ only represents the harmonic function h with $\lim_{|x|\to\infty} h(x) = 0$. When $D = R^d$, we also omit R^d i.e., $G(x, y) = c_d/|x-y|^{d-2}$.

The following crucial inequality called the $3G$ estimate (see [Z1] when $D_r = B_r$) can be proved by the same elementary argument as that in [Z1] when $D_r = B_r^*$.

Proposition 4. *For any $r > 0$ and any x, y, z in D_r, we have*

$$\frac{G_{D_r}(x, y)G_{D_r}(y, z)}{G_{D_r}(x, z)} \leq C(d)(|x-y|^{2-d} + |y-z|^{2-d}), \tag{14}$$

where

$$C(d) = 2^{d+3}(d-2)^2 c_d. \tag{15}$$

We end this section with a definition of nonnegativity of the operator H.

Definition 2. For $V \in K_d^{\text{loc}}$, $H \equiv -\Delta/2 + V \geq 0$ iff for any $\varphi \in C_c^\infty(R^d)$,

$$\frac{1}{2}\int_{R^d} \nabla\varphi \cdot \nabla\varphi\, dx + \int_{R^d} V\varphi^2\, dx \geq 0, \tag{16}$$

where $C_c^\infty(R^d) = \{$all infinitely differentiable functions in R^d with a compact support$\}$.

Remark. This is equivalent to the condition: Spectrum of $H \subseteq [0, \infty)$.

3. Preliminary lemmas and propositions

We start with a probabilistic characterization of nonnegativity of H. Recall the notation $e_V(\cdot)$ given in (10) and define the exit time of a Markov process X from a domain D by

$$\begin{aligned} \tau_D &= \inf[t < \xi : X_t \notin D], && \text{if the set in the bracket is nonempty}, \\ &= \xi, && \text{if the set in the bracket is empty}, \end{aligned}$$

where ξ is the lifetime of X.

Theorem 1. *Let $V \in K_d^{\text{loc}}$. Then $H = -\Delta/2 + V \geq 0$ if and only if for any $r > 0$,*

$$E^x[e_V(\tau_{B_r})] \not\equiv \infty \quad \textit{in } B_r. \tag{17}$$

Proof. If $-\Delta/2 + V \geq 0$ then by the Allegretto-Piepenbrink theory (see Theorem C.8.1 in [S2] or Theorem 5 in [Z4]), there exists a positive and continuous solution $u > 0$ of $Hu = 0$ in R^d. For any $r > 0$, we have $\inf_{x\in B_r} u(x) \geq \inf_{x\in\overline{B_r}} u(x) > 0$. Thus by Theorem 4 in [Z4], (17) holds.

Conversely, if (17) holds for each $r > 0$, then by Theorem 4 in [Z4], all eigenvalues of $-\Delta/2 + V$ in B_r are larger than zero, i.e.,

$$\text{Spec}[(-\Delta/2 + V)|_{B_r}] \subseteq (0, \infty). \tag{18}$$

For any φ in $C_c^\infty(R^d)$, we can find $r > 0$ such that $\text{supp}(\varphi) \subseteq B_r$. Then by (18) we have

$$\frac{1}{2}\int_{R^3} \nabla\varphi \cdot \nabla\varphi\, dx + \int_{R^d} V\varphi^2\, dx = \frac{1}{2}\int_{B_r} \nabla\varphi \cdot \nabla\varphi\, dx + \int_{R^d} V\varphi^2\, dx \geq 0.$$

This shows $H = -\Delta/2 + V \geq 0$ by the Definition 2. □

Proposition 5. *Suppose $V \in K_d^{\text{loc}}$ and $H = -\Delta/2 + V \geq 0$. Then for any bounded C^2 domain D in R^d, there exists a jointly continuous Poisson kernel*

$K_D^V(x, z) > 0$, $(x, z) \in D \times \partial D$ *for the operator* H, *i.e., for any* $f \in C(\partial D)$ *and* $x \in D$, *we have*

$$\int_{\partial D} K_D^V(x, z) f(z) \sigma(dz) = E^x[e_V(\tau_D) f(X(\tau_D))]. \tag{19}$$

Proof. The proposition follows from Theorem 1 and Corollary 2 in [Z3]. □

We now state a crucial lemma in a general situation, which can be proved by the same argument as that in [AS, Theorem 1.2].

Lemma 1 (Khasminskii's lemma). *Suppose* D *is a domain in* R^d, $V \in \mathscr{B}(D)$, $\{X_t\}$ *is a Markov process in* D *and* τ *is either a constant time or an exit time. If*

$$\sup_{x \in D} \mathbb{E}^x \left[\int_0^\tau |V(X_s)|\, ds \right] \le \alpha < 1, \tag{20}$$

then

$$\sup_{x \in D} \mathbb{E}^x \left[\exp\left(\int_0^\tau |V(X_s)|\, ds \right) \right] \le \frac{1}{1-\alpha}. \tag{21}$$

Note that we use $\mathbb{E}^x$ for the general process.

Lemma 2. *Under the same assumptions as in Lemma* 1, *let* $x \in D$, $A \in \mathscr{F}$ *with* $\mathbb{P}^x(A) \ge c_1 > 0$, *and* $\mathbb{E}^x[\int_0^\tau |V(X_s)|\, ds] \le c_2$.

Then we have

$$\mathbb{E}^x[A; e_V(\tau)] \ge c_1 \exp(-c_2/c_1). \tag{22}$$

Proof. It is easily done by using the Jensen's inequality for $\mathbb{E}^x(\cdot | A)$. □

Lemma 3. *Let* D *be a ball* $B = B(w, l)$ *in* R^d *or* B_r^* $(r > 0)$ *or* R^d *itself, and let* U *be a subdomain of* D *satisfying*

$$|||\, 1_U V \,||| < \frac{1}{2C(d)}, \tag{23}$$

where $C(d)$ *is given in* (15). *Then for all* x *and* y *in* D *with* $x \neq y$ *we have*

$$E_{y,D}^x \left[\exp\left(\int_0^{\tau_U} |V(X_s)|\, ds \right) \right] \le \frac{1}{1 - 2C(d)\, |||\, 1_U V \,|||}. \tag{24}$$

Proof. For each $x, y \in D$ with $x \neq y$, by Proposition 4 and Fubini's theorem we have

$$\begin{aligned} E_{y,D}^x \left[\int_0^{\tau_U} |V(X_s)|\, ds \right] &= \int_0^\infty E_{y,D}^x [s < \tau_U; |V(X_s)|]\, ds \\ &= G_D(x, y)^{-1} \int_0^\infty E^x[s < \tau_U; G_D(X_s, y)|V(X_s)|]\, ds \\ &\le G_D(x, y)^{-1} \int_U G_D(x, w) G_D(w, y) |V(w)|\, dw \\ &\le C(d) \left[\int_U \frac{|V(w)|}{|x-w|^{d-2}}\, dw + \int_U \frac{|V(w)|}{|y-w|^{d-2}}\, ds \right] \\ &\le 2C(d)\, |||\, 1_U V \,|||. \end{aligned} \tag{25}$$

Thus (24) follows from Lemma 1 and (25). □

Now we assume $V \in K_d^\infty$ $(d \geq 3)$. By Definition 1, there exists a number $A > 0$ such that

$$\text{(26)} \qquad ||| \, 1_{B_A^*} V \, ||| = \sup_{x \in R^d} \int_{|y|>A} \frac{|V(y)|}{|y-x|^{d-2}} \, dy < \frac{1}{4C(d)} .$$

Since $K_d^\infty \subseteq K_d$ (Proposition 1), by the definition of K_d given in (6), there exists a number a with $0 < a \leq A$ such that for any ball $B(w, r)$ with $0 < r \leq a$,

$$\text{(27)} \qquad ||| \, 1_{B(w,r)} V \, ||| = \sup_{x \in R^d} \int_{B(w,r)} \frac{|V(y)|}{|y-x|^{d-2}} \, dy < \frac{1}{4C(d)} .$$

Fix the two numbers A and a which depend on d and V only.

In the following, we assume

$$\text{(28)} \qquad \sup_{x \in B_{2A}} E^x[e_V(\tau_{B_{2A}})] < \infty .$$

The next lemma follows from Lemma 1 and (26).

Lemma 4.

$$\sup_{x \in R^d} E^x \left[\exp \left(\int_0^{\tau_{B_A^*}} |V(X_s)| \, ds \right) \right] \leq 2 .$$

Put

$$\text{(29)} \qquad u_0(x) = E^x[e_V(\infty)] = E^x \left[\exp \left(- \int_0^\infty V(X_s) \, ds \right) \right] , \qquad x \in R^d .$$

Theorem 2. *$u_0(x) \not\equiv \infty$ in R^d if and only if $u_0(x)$ is bounded in R^d. If so, then u_0 is a continuous solution of $Hu = 0$ in R^d.*

Proof. Since for any ball B and $x \in B$, we have

$$\begin{aligned} u_0(x) &= E^x[e_V(\tau_B) E^{X(\tau_B)}[e_V(\infty)]] \\ &= E^x[e_V(\tau_B) u_0(X(\tau_B))] . \end{aligned}$$

Then by an estimate given by Theorem 1 in [Z1] and a chain argument, it shows that if there is one point $x_0 \in R^d$ with $u_0(x_0) < \infty$ then u_0 is bounded on any compact set, in particular we have $M = \sup_{|x| \leq A} u_0(x) < \infty$. Now for each $x \in B_A^*$,

$$\text{(30)} \qquad \begin{aligned} u_0(x) &= E^x[\tau_{B_A^*} = \infty , e_V(\tau_{B_A^*})] \\ &\quad + E^x[\tau_{B_A^*} < \infty ; e_V(\tau_{B_A^*}) u_0(X(\tau_{B_A^*}))] \\ &\leq (M+1) E^x[e_V(\tau_{B_A^*})] \leq 2(M+1) . \end{aligned}$$

The last inequality is due to Lemma 4. This shows that u_0 is bounded in R^d.

Now suppose that $C \equiv \sup_{x \in R^d} u_0(x) < \infty$. Then we have

$$\text{(31)} \qquad \int_{R^d} G(x, y) |V(y)| u_0(y) \, dy \leq C c_d ||| \, V \, ||| < \infty .$$

By the Fubini's theorem together with (31) and the piecewise integration, we get

$$\text{(32)} \qquad \begin{aligned} \int_{R^d} G(x, y) V(y) u_0(y) \, dy &= E^x \left[\int_0^\infty V(X_t) E^{X(t)}[e_V(\infty)] \, dt \right] \\ &= E^x \left\{ \int_0^\infty V(X_t) \exp \left[- \int_t^\infty V(X_s) \, ds \right] \right\} \\ &= E^x[1 - e_V(\infty)] = 1 - u_0(x) . \end{aligned}$$

Applying $\Delta/2$ on both sides of (32) we obtain $-\Delta/2u_0(x) + V(x)u_0(x) = 0$, i.e., u_0 is a distributional solution of $Hu = 0$. By Proposition 3(ii), $\{G(x, y)|V(y)|u_0(y): x \in R^d\}$ is uniformly integrable. So the continuity of $u_0(x)$ follows from (32). □

Lemma 5. *There exists a joint continuous Poisson kernel* $K^V_{B^*_A}(x, z) > 0$ *for* H *on* $B^*_A \times \partial B^*_A$ *with the estimate*:

$$K^V_{B^*_A}(x, z) \le 2(d-2)c_d \frac{|x|^2 - A^2}{A|x-z|^d}. \tag{33}$$

Remark. Poisson kernel here is for representing the solution u of $Hu = 0$ with $\lim_{|x|\to\infty} u(x) = 0$.

Proof. By (24) in Lemma 3 with $D = U = B^*_A$ and condition (26) we have

$$\sup_{(x, y)\in B^*_A\times\overline{B}^*_A} E^x_{y, B^*_A}[e_V(\tau_{B^*_A})] \le 2. \tag{34}$$

By Proposition 3(ii), (34) and the method similar to that in [Z3], we have that $(x, z) \to E^x_{z, B^*_A}[e_V(\tau_{B^*_A})]$ is joint continuous on $B^*_A \times \partial B^*_A$ and

$$K^V_{B^*_A}(x, z)/K_{B^*_A}(x, z) = E^x_{z, B^*_A}[e_V(\tau_{B^*_A})]. \tag{35}$$

Thus (33) follows from (13), (34) and (35). □

For each r in $(2A, \infty]$, put $J_{A, r} = \{y \in R^d : A < |y| < r\}$ (note: $J_{A,\infty} = B^*_A$). We define a stopping time:

$$\sigma_r = \tau_{B_{2A}} + \tau_{J_{A,r}} \circ \theta_{\tau_{B_{2A}}}, \tag{36}$$

and an operator in the Banach space $C(\partial B_A)$:

$$S_{V, r}f(z) = E^z[\sigma_r < \infty \text{ and } X(\sigma_r) \in \partial B_A; e_V(\sigma_r)f(X(\sigma_r))], \quad z \in \partial B_A, \ f \in C(\partial B_A). \tag{37}$$

Lemma 6. *For each* $r \in (2A, \infty]$, $S_{V, r}$ *is an integral operator in* $C(\partial B_A)$ *with a continuous kernel* $\Phi_r(\cdot, \cdot)$:

$$S_{V, r}f(z) = \int_{\partial B_A} \Phi_r(z, w)\sigma(dw), \tag{38}$$

where

$$\Phi_r(z, w) = \int_{\partial B_{2A}} K^V_{B_{2A}}(z, u)K^V_{J_{A,r}}(u, w)\sigma(du), \quad (z, w) \in \partial B_A \times \partial B_A, \tag{39}$$

and $\sigma(\cdot)$ *is the area measure.*

Proof. In the following, for any closed set Λ in R^d, T_Λ denotes the first hitting time on Λ. By the strong Markov property, we have for any $f \in C(\partial B_A)$,

$$\begin{aligned} S_{V, r}f(z) &= E^z\{e_V(\tau_{B_{2A}})E^{X(\tau_{B_{2A}})}[T(\partial B_A) < T(\partial B_r), \\ &\qquad\qquad e_V(T(\partial B_A))f(X(T(\partial B_A)))]\} \\ &= \int_{\partial B_{2A}} K^V_{B_{2A}}(z, u)\left[\int_{\partial B_A} K^V_{J_{A,r}}(u, w)f(w)\sigma(dw)\right]\sigma(du) \\ &= \int_{\partial B_A}\left[\int_{\partial B_{2A}} K^V_{B_{2A}}(z, u)K^V_{J_{A,r}}(u, w)\sigma(du)\right] f(w)\sigma(dw). \end{aligned}$$

This proves (39). $K^V_{B_{2A}}(\cdot\,,\,\cdot)$ and $K^V_{J_{A,r}}(\cdot\,,\,\cdot)$ $(r < \infty)$ are continuous by Proposition 5. The continuity of $K^V_{J_{A,\infty}} = K^V_{B^*_A}$ is given by Lemma 5. So the kernel Φ_r given in (39) is continuous. □

Lemma 7. $\sup_{(z,w)\in\partial B_A\times\partial B_A}|\Phi_r(z,w)-\Phi_\infty(z,w)| \to 0$ *as* $r\uparrow\infty$.

Proof. For $r\in(2A,\infty)$, by the strong Markov property, we have for all $u\in\partial B_{2A}$, and $w\in\partial B_A$,

$$(40)\qquad K^V_{B^*_A}(u,w) = K^V_{J_{A,r}}(u,w) + \int_{\partial B_r} K^V_{J_{A,r}}(u,y)K^V_{B^*_A}(y,w)\sigma(dy).$$

By the estimate (33) for $K^V_{B^*_A}$ in Lemma 5,

$$(41)\ |K^V_{B^*_A}(u,w)-K^V_{J_{A,r}}(u,w)| \le 2(d-2)C_d\frac{r^2-A^2}{A(r-A)^d}\int_{\partial B_r}K^V_{J_{A,r}}(u,y)\sigma(dy).$$

By Lemma 4,

$$(42)\qquad \begin{aligned}\int_{\partial B_r}K^V_{J_{A,r}}(u,y)\sigma(dy) &= E^u[T(\partial B_r)<T(\partial B_A);\,e_V(T(\partial B_r))]\\ &\le E^u\left[\exp\left(\int_0^{\tau_{B^*_A}}|V(X_s)|\,ds\right)\right]\le 2.\end{aligned}$$

Then by (39)–(42) we obtain

$$(43)\quad |\Phi_r(z,w)-\Phi_\infty(z,w)| \le 4(d-2)c_d\frac{r^2-A^2}{A(r-A)^d}\int_{\partial B_{2A}}K^V_{B_{2A}}(z,w)\sigma(du).$$

Since $\int_{\partial B_{2A}}K^V_{B_{2A}}(z,u)\sigma(du) = E^z[e_V(\tau_{B_{2A}})]$, the lemma follows from (43) and the assumption (28). □

We need some elementary properties for the integral operators in the Banach space $C(U)$, where U is any compact set with a finite measure φ on it. For any continuous function $Q(\cdot\,,\,\cdot)>0$ on $U\times U$ we define an operator in $C(U)$:

$$(44)\qquad M_Qf(\cdot) = \int_U Q(\cdot\,,y)f(y)\mu(dy),$$

and put $\lambda_Q = \lim_{n\to\infty}\sqrt[n]{\|(M_Q)^n\|}>0$.

Lemma 8. *If* $Q_1>Q_2>0$ *on* $U\times U$, *then* $\lambda_{Q_1}>\lambda_{Q_2}$.

Proof. Since $U\times U$ is compact, there exists an $\varepsilon>0$ such that $Q_1>(1+\varepsilon)Q_2$ on $U\times U$. Obviously, λ_Q is nondecreasing with Q. Thus we have $\lambda_{Q_1}\ge\lambda_{(1+\varepsilon)Q_2}=(1+\varepsilon)\lambda_{Q_2}>\lambda_{Q_2}$. □

Lemma 9. *If* $\sup_{(x,y)\in U\times U}|Q_n(x,y)-Q(x,y)|\to 0$ *as* $n\to\infty$, *then* $\lambda_{Q_n}\to\lambda_Q$ *as* $n\to\infty$.

Proof. Since $Q>0$ in $U\times U$, for any $\varepsilon>0$, there exists $N\ge 1$ such that for all $n\ge N$, $x,y\in U$, $(1-\varepsilon)Q(x,y)\le Q_n(x,y)\le(1+\varepsilon)Q(x,y)$. Then $(1-\varepsilon)\lambda_Q=\lambda_{(1-\varepsilon)Q}\le\lambda_{Q_n}\le\lambda_{(1+\varepsilon)Q}=(1+\varepsilon)\lambda_Q$. □

Now for the compact set ∂B_A, $r\in(2A,\infty]$. Put

$$(45)\qquad \lambda_r(V) = \lim_{n\to\infty}\sqrt[n]{\|(S_{V,r})^n\|},$$

where $S_{V,r}$ is defined by (37) and is an integral operator in $C(\partial B_A)$ with a strictly positive and continuous kernel Φ_r by Lemma 6.

The following lemma is a key step for the main theorem.

Lemma 10. *For* $r \in (2A, \infty]$, $E^x[e_V(\tau_{B_r})] \not\equiv \infty$ *if and only if* $\lambda_r(V) < 1$.

Proof. Put $f_r(z) = E^z[X(\sigma_r) \in \partial B_r; e_V(\sigma_r)]$, if $r < \infty$, and

$$f_\infty(z) = E^z[\sigma_r = \infty; e_V(\sigma_r)].$$

Then for $z \in \partial B_A$, by the strong Markov property and the transient property of $\{X_t\}$ in R^d $(d \geq 3)$ for the case $r = \infty$, we have for each $r \in (2A, \infty]$,

$$E^z[e_V(\tau_{B_r})] = \sum_{n=0}^{\infty} (S_{V,r})^n f_r(z). \tag{46}$$

If $\lambda_r(V) < 1$, then by (45), the series in (46) converges uniformly on ∂B_A, we have $E^z[e_V(\tau_{B_r})] < \infty$. Conversely, if $E^x[e_V(\tau_{B_r})] \not\equiv \infty$, then by Theorem 7 in [Z1] when $r < \infty$ and by Theorem 2 when $r = \infty$, $x \to E^x[e_V(\tau_{B_r})]$ is a bounded continuous function in ∂B_A. Then by Dini's theorem, the convergence in the series in (46) is uniform on ∂B_A. Since $f_r(z)$ is a strictly positive and continuous function on ∂B_A, $c = \min_{z \in \partial B_A} f_r(z) > 0$. By the uniform convergence we can find an integer N such that $\|(S_{V,r})^N f_r\| < c$. Therefore $\|(S_{V,r})^N\| = \|(S_{V,r})^N 1\| \leq c^{-1} \|(S_{V,r})^N f_r\| < 1$, and then $\sqrt[N]{\|(S_{V,r})^N\|} < 1$. Thus we obtain $\lambda_V(V) = \inf_n \sqrt[n]{\|(S_{V,r})^n\|} < 1$. □

Lemma 11.

$$\lim_{r \to \infty} \lambda_r(V) = \lambda_\infty(V). \tag{47}$$

Proof. (47) follows from Lemma 7 and Lemma 9. □

Lemma 12. *For each* $r \in (2A, \infty)$,

$$\lambda_r(V) < \lambda_\infty(V). \tag{48}$$

Proof. Noticing $B_A^* = J_{A,\infty}$ we have by (39) and (40), for $2A < r < \infty$ and any $z, w \in \partial B_A$,

$$\begin{aligned}\Phi_\infty(z, w) &- \Phi_r(z, w) \\ &= \int_{\partial B_{2A}} \int_{\partial B_r} K^V_{B_{2A}}(z, u) K^V_{J_{A,r}}(u, y) K^V_{B_A^*}(y, w) \sigma(du) \sigma(dy) > 0.\end{aligned}$$

Thus (48) follows from Lemma 8. □

Lemma 13. $H \equiv -\Delta/2 + V \geq 0$ *if and only if* $\lambda_\infty(V) \leq 1$.

Proof. By Theorem 1 and Lemma 10, we have that $H \geq 0$ if and only if $\lambda_r(V) < 1$ for $2A < r < \infty$. By Lemma 11 and Lemma 12, the latter condition is equivalent to $\lambda_\infty(V) \leq 1$.

Lemma 14. *Suppose* $\{V_n\} \subseteq K_d^\infty$ *and* $\lim_{n \to \infty} |||\, V_n - V \,||| = 0$. *Then for sufficiently large* n, V_n *satisfies the condition* (26)–(28) *and*

$$\lim_{n \to \infty} \lambda_\infty(V_n) = \lambda_\infty(V). \tag{49}$$

Proof. Obviously, (26) and (27) hold for V_n when n is sufficiently large. Since (28) holds for V, by Theorem 4 in [Z4] and Lemma A.4.3 in [A-S], there is $\beta > 1$ such that

$$\sup_{x\in B_{2A}} E^x[e_{\beta V}(\tau_{B_{2A}})] < \infty . \tag{50}$$

By Lemma 1, it is easy to see that for sufficiently large n,

$$\sup_{x\in B_{2A}} E^x[e_{\beta(V_n-V)/(\beta-1)}(\tau_{B_{2A}})] < \infty . \tag{51}$$

Thus, for $x \in B_{2A}$, by Hölder's inequality, (50) and (51)

$$\begin{aligned} E^x[e_{Vn}(\tau_{B_{2A}})] &= E^x[e_V(\tau_{B_{2A}})e_{V_n-V}(\tau_{B_{2A}})] \\ &\le \{E^x[e_{\beta V}(\tau_{B_{2A}})]\}^{1/\beta}\{E^x[e_{\beta(V_n-V)/(\beta-1)}(\tau_{B_{2A}})]\}^{(\beta-1)/\beta} < \infty , \end{aligned}$$

i.e., (28) holds for V_n.

In order to prove (49), let $\Phi^n_\infty(z, w) = \int_{\partial B_{2A}} K^{V_n}_{B_{2A}}(z, u)K^{V_n}_{B^*_A}(u, w)\sigma(du)$. By Lemma 9, we need only prove that

$$\sup_{(z,w)\in\partial B_A\times\partial B_A} |\Phi^n_\infty(z, w) - \Phi_\infty(z, w)| \to 0 \quad \text{as } n \to \infty . \tag{52}$$

By (39), it suffices to prove that

$$\sup_{(z,u)\in\partial B_A\times\partial B_{2A}} |K^{V_n}_{B_{2A}}(z, u) - K^{V}_{B_{2A}}(z, u)| \to 0, \tag{53}$$

and

$$\sup_{(u,w)\in\partial B_{2A}\times\partial B_A} |K^{V_n}_{B^*_A}(u, w) - K^{V}_{B^*_A}(u, w)| \to 0, \tag{54}$$

as $n \to \infty$. By (50) and the conditioned gauge theorem (see Theorem 1 in [Z2]), we have

$$M \equiv \sup_{(z,u)\in B_{2A}\times\partial B_{2A}} E^z_{u, B_{2A}}[e_{\beta V}(\tau_{B_{2A}})] < \infty .$$

Take an integer $N \ge \beta/(\beta - 1)$, then by Hölder's inequality,

$$\begin{aligned} &|E^z_{u, B_{2A}}[e_{V_n}(\tau_{B_{2A}})] - E^z_{u, B_{2A}}[e_V(\tau_{B_{2A}})]| \\ &\quad = |E^z_{u, B_{2A}}[e_V(\tau_{B_{2A}})(e_{V_n-V}(\tau_{B_{2A}}) - 1)]| \\ &\quad \le M^{1/\beta}\left\{E^z_{u, B_{2A}}\left[\exp\int_0^{\tau_{B_{2A}}}|(V_n - V)(X_s)|\,ds - 1\right]^N\right\}^{1/N} . \end{aligned} \tag{55}$$

By a simple inequality: for $N \ge 1$ and $x > 0$.

$$(e^x - 1)^N \le \sum_{k=0}^N \binom{N}{k}(e^{kx} - 1),$$

and Lemma 3,

$$\begin{aligned} &E^z_{u, B_{2A}}\left[\exp\int_0^{\tau_{B_{2A}}}|(V_n - V)(X_s)|\,ds - 1\right]^N \\ &\quad \le \sum_{k=0}^N \binom{N}{k}\left[\frac{1}{1 - 2kC(d)\,|||\, V_n - V\,|||} - 1\right] . \end{aligned} \tag{56}$$

For $(z, u) \in \partial B_A \times \partial B_{2A}$, by (13),

$$K_{B_{2A}}(z, u) = (d-2)c_d \frac{4A^2 - |z|^2}{2A|z-u|^d} \leq 2(d-2)c_d A^{1-d}. \tag{57}$$

Let W be one of V and V_n, $n \geq 1$, we have by Corollary 2 in [Z3],

$$K^W_{B_{2A}}(z, u) = K_{B_{2A}}(z, u) E^x_{z, B_{2A}}[e_W(\tau_{B_{2A}})]. \tag{58}$$

Since $||| V - V_n ||| \to 0$ as $n \to \infty$, (53) follows from (55)–(58).

(54) follows in a similar and easier way. □

Lemma 15. *If $V_1 \leq V_2$ and $V_1 \not\equiv V_2$ a.e. in R^d, then*

$$\lambda_\infty(V_1) > \lambda_\infty(V_2). \tag{59}$$

Proof. Let

$$\Phi^i_\infty(z, w) = \int_{\partial B_{2A}} K^{V_i}_{B_{2A}}(z, u) K^{V_i}_{B^*_A}(u, w)\sigma(du) \qquad (i = 1, 2). \tag{60}$$

By Lemma 8, we need only prove that for each $(z, w) \in \partial B_A \times \partial B_A$,

$$\Phi^1_\infty(z, w) > \Phi^2_\infty(z, w). \tag{61}$$

Since $V_1 \leq V_2$, we have by (35) and (58),

$$K^{V_1}_{B_{2A}}(z, u) \geq K^{V_2}_{B_{2A}}(z, u) \quad \text{for each } (z, u) \in \partial B_A \times \partial B_{2A}, \tag{62}$$

and

$$K^{V_1}_{B^*_A}(u, w) \geq K^{V_2}_{B^*_A}(u, w) \quad \text{for each } (u, w) \in \partial B_{2A} \times \partial B_A. \tag{63}$$

Since $V_1 \not\equiv V_2$ a.e. in R^d, we have either $V_1 \not\equiv V_2$ a.e. in B_{2A} or $V_1 \not\equiv V_2$ a.e. in B_{A^*}. In the former case

$$E^z_{u, B_{2A}}\left[-\int_0^{\tau_{B_{2A}}} V_1(X_t)\,dt + \int_0^{\tau_{B_{2A}}} V_2(X_t)\,dt\right] = \int_{B_{2A}} \frac{G_{B_{2A}}(z, y)[V_2(y) - V_1(y)]K_{B_{2A}}(y, u)}{K_{B_{2A}}(z, u)}\,dy > 0.$$

Therefore by $V_1 \leq V_2$ in R^d we have

$$E^z_{u, B_{2A}}[e_{V_1}(\tau_{B_{2A}})] > E^z_{u, B_{2A}}[e_{V_2}(\tau_{B_{2A}})]. \tag{64}$$

This together with (58) shows that

$$K^{V_1}_{B_{2A}}(z, u) > K^{V_2}_{B_{2A}}(z, u). \tag{65}$$

So (61) follows from (60), (63), and (65). Similarly in the latter case, we have

$$K^{V_1}_{B^*_A}(u, w) > K^{V_2}_{B^*_A}(u; w). \tag{66}$$

Then (61) follows from (60), (62), and (66). □

Lemma 16. *For all $z \in \partial B_{4A}$ and $y \in B^*_{6A}$, we have*

$$e^{-1} \leq E^z[\tau_{B^*_A} = \infty; e_V(\tau_{B^*_A})], \tag{67}$$

and

$$e^{-8} \leq E^z_y[\xi < \tau_{B^*_A}; e_V(\xi)]. \tag{68}$$

(*Note that* E_y^z *denotes* E_{y,R^d}^z *and* ξ *is the lifetime of the conditional* y*-process.*)

Proof. For $z \in \partial B_{4A}$,

$$P^z(\tau_{B_A^*} = \infty) = 1 - \frac{A^{d-2}}{|z|^{d-2}} = 1 - 4^{2-d} \geq \frac{3}{4},$$

and by (26),

$$E^z\left[\int_0^{\tau_{B_A^*}} |V(X_s)|\,ds\right] \leq c_d \int_{|y|>A} \frac{|V(y)|}{|z-y|^{d-2}}\,dy \leq 2^{-5}.$$

Then (67) follows from Lemma 2 and a simple inequality: $\frac{3}{4}e^{-4/3}2^{-5} \geq e^{-1}$.

Now for $z \in \partial B_{4A}$ and $y \in B_{6A}^*$, by (13) with $D_A = B_A^*$,

$$\begin{aligned}
P_y^z(\tau_{B_A^*} < \xi) &= |z-y|^{d-2}E^z(\tau_{B_A^*} < \infty;\, |X(\tau_{B_A^*}) - y|^{2-d}) \\
&= (d-2)c_d|z-y|^{d-2}\int_{\partial B_A} \frac{|z|^2 - A^2}{A|z-w|^d|w-y|^{d-2}}\sigma(dw) \\
&= 15(d-2)c_dA\int_{\partial B_A} \frac{1}{|z-w|^2}\left(\frac{|z-y|}{|z-w|\,|w-y|}\right)^{d-2}\sigma(dw) \\
&\leq 15(d-2)c_dA\int_{\partial B_A} \frac{1}{(3A)^2}(|z-w|^{-1} + |w-y|^{-1})^{d-2}\sigma(dw) \\
&\leq \frac{5}{3}(d-2)c_dA^{-1}\int_{\partial B_A}\left(\frac{1}{3A} + \frac{1}{5A}\right)^{d-2}\sigma(dw) \\
&= \frac{5}{3}\left(\frac{8}{15}\right)^{d-2} \leq \frac{8}{9}.
\end{aligned}$$

So $P_y^z(\xi < \tau_{B_A^*}) \geq 1 - \frac{8}{9} = \frac{1}{9}$ and by (26),

$$\begin{aligned}
E_y^z\left[\int_0^{\tau_{B_A^*}} |V(X_s)|\,ds\right] &\leq c_d\int_{|w|\geq A} \frac{|z-y|^{d-2}|V(w)|}{|z-w|^{d-2}|w-y|^{d-2}}\,dw \\
&\leq 2^{d-2}c_d \sup_{|x|\geq A}\int_{|w|\geq A} \frac{|V(w)|}{|x-w|^{d-2}}\,dw < \frac{1}{2}.
\end{aligned}$$

Thus (68) follows from Lemma 2 and a simple inequality $\frac{1}{9}e^{-9/2} \geq e^{-8}$. □

By similar arguments as those in Lemma 16, we have

Lemma 17. *Let* $0 < r \leq a$, *where* a *is given in* (27). *Then for any ball* $B = B(w, r)$ $(w \in R^d)$, $z \in \partial B(w, r/5)$ *and* $y \in B(w, r/10)$, *we have*

(69) $$e^{-3} \leq E_y^z[\xi = \tau_B;\, e_V(\tau_B)].$$

For $V \in K_d^\infty$, put

(70) $$u_0(x, y) = E_y^x[e_V(\xi)] \equiv E_y^x\left[\exp\left(-\int_0^\xi V(X_s)\,ds\right)\right], \qquad x \neq y,$$

where ξ is the life time of the y-process. Note E_y^x denotes E_{y,R^d}^x.

Lemma 18. *There exists a function* Q *on* $\overline{B_{4A}}$ *with* $0 < Q \le \infty$ *such that for each* $x \in \overline{B_{4A}}$,

$$e^{-1}Q(x) \le u_0(x) \le 2Q(x), \tag{71}$$

and for each $x \in \overline{B_{4A}}$, $y \in B_{6A}^*$,

$$e^{-8}5^{2-d}Q(x) \le u_0(x, y) \le 5^{d-1}Q(x). \tag{72}$$

Proof. For $x \in \overline{B_{4A}}$ and $y \in B_{6A}^*$, define a sequence of stopping times under P^x and P_y^x, respectively, $T_0 = 0$, for $n \ge 1$:

$$T_{2n-1} = T_{2n-2} + \tau_{B_{4A}} \circ \theta_{T_{2n-2}} \quad \text{on} \begin{cases} (T_{2n-2} < \infty) & \text{under } P^x, \\ (T_{2n-2} < \xi) & \text{under } P_y^x. \end{cases}$$

$$T_{2n} = T_{2n-1} + \tau_{B_A^*} \circ \theta_{T_{2n-1}}.$$

Put

$$Q(x) = \sum_{n=1}^{\infty} E^x[T_{2n-2} < \infty; e_V(T_{2n-1})].$$

By the strong Markov property, we have for $x \in \overline{B_{4A}}$,

$$\begin{aligned} u_0(x) &= \sum_{n=1}^{\infty} E^x[T_{2n-2} < \infty \text{ and } T_{2n} = \infty; e_V(\infty)] \\ &= \sum_{n=1}^{\infty} E^x\{T_{2n-2} < \infty; e_V(T_{2n-1})E^{X(T_{2n-1})}[\tau_{B_A^*} = \infty, e_V(\tau_{B_A^*})]\}. \end{aligned} \tag{73}$$

Then by Lemma 4 and Lemma 16 (67) we get (71) from (73).

Now for $x \in \overline{B_{4A}}$ and $y \in B_{6A}^*$, by the strong Markov property of the y-process, we have

$$u_0(x, y) = \sum_{n=1}^{\infty} E_y^x\{T_{2n-2} < \xi; e_V(T_{2n-1})E_y^{X(T_{2n-1})}[\xi < \tau_{B_A^*}; e_V(\xi)]\}. \tag{74}$$

For each $n \ge 1$, $X(T_{2n-1}) \in \partial B_{4A}$, on $(T_{2n-2} < \xi)$, then by Lemma 16 (68), Lemma 3 and (26), we obtain

$$\begin{aligned} e^{-8} &\le E_y^{X(T_{2n-1})}[\xi < \tau_{B_A^*}; e_V(\xi)] \\ &\le E_y^{X(T_{2n-1})}\left[\exp\left(\int_0^{\tau_{B_A^*}} |V(X_s)|\, ds\right)\right] \le 2, \end{aligned} \tag{75}$$

and

$$\begin{aligned} &E_y^x\{T_{2n-2} < \xi; e_V(T_{2n-1})\} \\ &\quad = |x - y|^{d-2}E^x\{T_{2n-2} < \infty; |X(T_{2n-1}) - y|^{2-d}e_V(T_{2n-1})\}. \end{aligned} \tag{76}$$

For $z \in \partial B_{4A}$, $x \in \overline{B}_{4A}$, and $y \in B_{6A}^*$,

$$\frac{1}{5} \le \frac{|x - y|}{|z - x| + |x - y|} \le \frac{|x - y|}{|z - y|} \le \frac{|x - z|}{|z - y|} + 1 \le 5. \tag{77}$$

Thus (72) follows from (74)–(77). □

Lemma 19. *There exists a constant* $c_1 > 0$ *depending on* d, A *and* a *only such that for each* $x \in \overline{B_{4A}}$, y *and* $y' \in \overline{B_{8A}}$ *with* $|y - x| \geq a$ *and* $|y' - x| \geq a$,

$$u_0(x, y) \leq c_1 u_0(x, y'). \tag{78}$$

Proof. Fix $x \in \overline{B_{4A}}$. For any $y \in \overline{B}_{8A}$ with $|y - x| \geq a$ and a ball $B = B(w, a/8)$ containing y. Set a sequence of stopping times:

$$S_0 = 0, \qquad S_{2n-1} = S_{2n-2} + \tau_{B^*(w, a/4)} \circ \theta_{S_{2n-2}},$$
$$S_{2n} = S_{2n-1} + \tau_{B(w, a)} \circ \theta_{S_{2n-1}} \qquad (n \geq 1).$$

Then

$$\begin{aligned} u_0(x, y) &= \sum_{n=1}^{\infty} E_y^x[\xi = S_{2n}; e_V(\xi)] \\ &= \sum_{n=1}^{\infty} E_y^x\{S_{2n-2} < \xi; e_V(S_{2n-1}) E_y^{X(S_{2n-1})}[\xi = \tau_{B(w, a)}; e_V(\xi)]\}. \end{aligned} \tag{79}$$

By (24) in Lemma 3, condition (27) and Lemma 17, we have for each $n \geq 1$, $(X(S_{2n-1}) \in \partial B(w, a/4))$,

$$e^{-3} \leq E_y^{X(S_{2n-1})}[\xi = \tau_{B(w, a)}; e_V(\tau_{B(w, a)})] \leq 2, \tag{80}$$

and

$$\begin{aligned} &E_y^x\{S_{2n-2} < \xi; e_V(S_{2n-1})\} \\ &\quad = |x - y|^{d-2} E^x\{S_{2n-2} < \infty; |X(S_{2n-1}) - y|^{2-d} e_V(S_{2n-1})\}. \end{aligned} \tag{81}$$

For each $z \in \partial B(w, a/4)$, we have

$$1 \leq \frac{a}{a/4 + a/8} \leq \frac{|x - y|}{|z - y|} \leq \frac{12A}{a/8} = 96\frac{A}{a}. \tag{82}$$

Thus it follows from (79) and (80) that

$$\begin{aligned} e^{-3} \sum_{n=1}^{\infty} E_y^x\{S_{2n-2} < \xi; e_V(S_{2n-1})\} &\leq u_0(x, y) \\ &\leq 2 \sum_{n=1}^{\infty} E_y^x\{S_{2n-2} < \xi; e_V(S_{2n-1})\}, \end{aligned} \tag{83}$$

and it follows from (81)–(83) that

$$\begin{aligned} e^{-3} \sum_{n=1}^{\infty} E^x\{S_{2n-2} < \xi; e_V(S_{2n-1})\} &\leq u_0(x, y) \\ &\leq 2 \left(\frac{96A}{a}\right)^{d-2} \sum_{n=1}^{\infty} E^x\{S_{2n-2} < \xi; e_V(S_{2n-1})\}. \end{aligned} \tag{84}$$

By (84), for any y and y' in $\overline{B_{8A}}$ with $|y - x| \geq a$ and $|y' - x| \geq a$, if y and y' are in the same ball $B(w, a/8)$, we have

$$u_0(x, y) \leq 2e^d (96A/a)^{d-2} u_0(x, y'). \tag{85}$$

Thus (78) follows from (85) and a ball chain argument. □

Lemma 20. *For any* $B = B(0, R)$, $0 < R < \infty$, *if* $G_B^V(x, y) > 0$ *exists for* $x, y \in B$, $x \neq y$, *then we have*

$$E_y^x[\xi < \tau_B; e_V(\xi)] \leq G_B^V(x, y)/G(x, y). \tag{86}$$

Proof. By [Z3, Theorem 7], the existence of G_B^V implies that

$$E_{y,B}^x[e_V(\xi)] = G_B^V(x, y)/G_B(x, y), \tag{87}$$

and there is $c > 0$ such that

$$\inf_{x, y \in B} E_{y,B}^x[e_V(\xi)] \geq c. \tag{88}$$

Fix x and y in B, $x \neq y$. Then for any $0 < r < \min(|x - y|, R - |x|, R - |y|)$, put $T_r =$ hitting time on $\partial B(y, r)$. We have $T_r \uparrow \xi$, a.s. as $r \downarrow 0$. For each such r, by (88),

$$E_{y,B}^x[e_V(\xi)|\mathscr{F}_{T_r}] = e_V(T_r)E_{y,B}^{X(T_r)}[e_V(\xi)] \geq ce_V(T_r).$$

Hence $\{e_V(T_r)\}$ are uniformly integrable as $r \downarrow 0$, then we have

$$\lim_{r \downarrow 0} E_{y,B}^x[e_V(T_r)] = E_{y,B}^x[e_V(\xi)]. \tag{89}$$

For each small r, we have

$$E_{y,B}^x[e_V(T_r)] = \frac{1}{G_B(x, y)} E^x[T_r < \tau_B; G_B(X(T_r), y)e_V(T_r)]. \tag{90}$$

Since $|X(T_r) - y| = r$,

$$\frac{c_d}{r^{d-2}} - \frac{c_d}{(R - |y|)^{d-2}} \leq G_B(X(T_r), y) \leq \frac{c_d}{r^{d-2}},$$

i.e.,

$$1 - \frac{r^{d-2}}{(R - |y|)^{d-2}} \leq \frac{G_B(X(T_r), y)}{G(X(T_r), y)} \leq 1. \tag{91}$$

By (89)–(91), we have

$$\frac{1}{G_B(x, y)} \lim_{r \downarrow 0} E^x[T_r < \tau_B; G(X(T_r), y)e_V(T_r)] = E_{y,B}^x[e_V(\xi)].$$

This is equivalent to

$$\frac{G(x, y)}{G_B(x, y)} \lim_{r \downarrow 0} E_y^x[T_r < \tau_B; e_V(T_r)] = E_{y,B}^x[e_V(\xi)].$$

By Fatou's lemma and (87), we obtain

$$\frac{G(x, y)}{G_B(x, y)} E_y^x[\xi < \tau_B; e_V(\xi)] \leq \frac{G_B^V(x, y)}{G_B(x, y)}.$$

This is the inequality (86). □

4. Main result

Let $V \in K_d^\infty$ $(d \geq 3)$ and put

$$H = -\frac{\Delta}{2} + V,$$

$$u_0(x) = E^x[e_V(\infty)] \equiv E^x\left[\exp\left(-\int_0^\infty V(X_s)\,ds\right)\right],$$

and

$$u_0(x, y) = E_y^x[e_V(\xi)] \equiv E_y^x \left[\exp\left(-\int_0^\xi V(X_s)\, ds \right) \right], \qquad x \neq y.$$

We choose $A > 0$ and $a > 0$ satisfying (26) and (27), respectively, and define the "shuttle operator": $S_V = S_{V,\infty}$, as in (37) $(r = \infty)$ and its spectral radius $\lambda(V) = \lambda_\infty(V)$, as in (45) $(r = \infty)$.

Main Theorem. *Let* $V \in K_d^\infty$ $(d \geq 3)$. *Then the following conditions are equivalent*:

(a) *There exists* $\varepsilon > 0$ *such that for any* $q \in K_d^\infty$ *with* $|||\, q \,||| < \varepsilon$, $H + q \geq 0$.
(b) *There exists* $\beta > 0$ *such that* $-\Delta/2 + (1 + \beta)V \geq 0$.
(c) *There exists a positive Green function* $G^V(\cdot, \cdot)$ *for* H.
(d) *There exists a solution* u *of* $Hu = 0$ *with* $\inf_x u(x) > 0$.
(e) $u_0(x) \not\equiv \infty$ *in* R^d.
(f) $u_0(x, y)$ *is bounded in* $R^d \times R^d \setminus \{(x, x) : x \in R^d\}$.
(g) $\lambda(V) < 1$.

Remark 1. We have other equivalent conditions in terms of small perturbations:
(a1) For any bounded Borel function q with compact support, there exists $\varepsilon > 0$ such that $H + \varepsilon q \geq 0$.
(a2) There exists a bounded Borel function q with compact support, $q \leq 0$ and $q \not\equiv 0$ a.e. in R^d such that $H + q \geq 0$.
(a3) There exists a function $q \in K_d^\infty$, $q \leq 0$, and $q \not\equiv 0$ a.e. in R^d such that $H + q \geq 0$.
and in terms of positive solutions:
(d1) There exists a continuous solution $u > 0$ of $Hu = 0$ with the limit: $\lim_{|x| \to \infty} u(x) > 0$.
(d2) There exists a solution u of $Hu = 0$ with $0 < \inf_x u(x) \leq \sup_x u(x) < \infty$.

Remark 2. By Theorem 2 (generalized "Gauge Theorem"), condition (e) is equivalent to
(e1) $u_0(x)$ is bounded in R^d.

Remark 3. We shall prove that condition (f) is equivalent to
(f1) $u_0(x_0, y_0) < \infty$ for some (x_0, y_0) in $R^d \times R^d$, $x_0 \neq y_0$.
This equivalence can be regarded as a generalized conditioned Gauge Theorem (see [C-F-Z] and [F]).

Proof of Main Theorem. Since every bounded Borel function with compact support is in K_d^∞, we have obvious implications: (a) $\Rightarrow$ (a1) $\Rightarrow$ (a2) $\Rightarrow$ (a3).

(a3) $\Rightarrow$ (g): Since $q \leq 0$, $H \geq H + q \geq 0$. By Theorem 1, (28) is satisfied for V and $V + q$. By Lemma 13, $H + q \geq 0$ implies $\lambda(V + q) \leq 1$. Then we obtain by Lemma 15 that $\lambda(V) < \lambda(V + q) \leq 1$, i.e., (g) holds.

(g) $\Rightarrow$ (a): Since $\lambda(V) < 1$, by Lemma 14, there exists $\varepsilon > 0$ such that if $|||\, q \,||| < \varepsilon$, then $\lambda(V + q) < 1$. Therefore we have $H + q = -\Delta/2 + (V + q) \geq 0$ by Lemma 13. Thus (a) and (g) are equivalent.

It follows from Lemma 10 with $r = \infty$ that (e) and (g) are equivalent.

(e) $\Rightarrow$ (d1): By the second statement of Theorem 2, $u_0 > 0$ is a continuous solution of $Hu = 0$. By the equality (32) and Proposition 3(ii) we have

$$\lim_{|x|\to\infty} u_0(x) = 1 - \lim_{|x|\to\infty} \int_{R^d} \frac{c_d V(y) u_0(y)}{|x-y|^{d-2}}\, dy$$
$$= 1 - c_d \int_{R^d} \left[\lim_{|x|\to\infty} \frac{V(y)u_0(y)}{|x-y|^{d-2}} \right] dy$$
$$= 1.$$

So (d1) holds.

(d) $\Rightarrow$ (e): Suppose there is a solution u of $Hu = 0$ with $u(x) \geq c > 0$ for all $x \in R^d$. Then for any $R > 0$, we have

$$u(0) = E^0[e_V(\tau_{B(0,R)}) u(X(\tau_{B(0,R)}))] \geq c E^0[e_V(\tau_{B(0,R)})].$$

By Fatou's lemma, we obtain

$$E^0[e_V(\infty)] \leq \lim_{R\to\infty} E^0[e_V(\tau_{B(0,R)})] \leq c^{-1} u(0) < \infty,$$

i.e., (e) holds.

Since we have (d1) $\Rightarrow$ (d2) $\Rightarrow$ (d) obviously, we proved (d) and (e) are equivalent.

(e) $\Rightarrow$ (f): By Theorem 2, we may suppose $M \equiv \sup_{x\in R^d} u_0(x) < \infty$. Then by (71) and (72) in Lemma 18 we have that for $x \in \overline{B_{4A}}$ and $y \in B^*_{6A}$,

$$u_0(x, y) \leq c_0 M, \tag{92}$$

where $c_0 = e5^{d-1}$. By (92) and (78) in Lemma 19 we have that for $x \in \overline{B_{4A}}$ and any y in $\overline{B_{6A}}$ with $|y - x| \geq a$,

$$u_0(x, y) \leq c_0 c_1 M. \tag{93}$$

Now for $x \in \overline{B_{2A}}$ and $0 < |y - x| < a$, by Lemma 3 and condition (27), we have

$$\begin{aligned} u_0(x, y) &= E^x_y\{\xi = \tau_{B(y,a)}; e_V(\xi)\} \\ &\quad + E^x_y\{\tau_{B(y,a)} < \xi; u_0(X(\tau_{B(y,a)}), y)\} \\ &\leq E^x_y\left\{\exp\left(\int_0^{\tau_{B(y,a)}} |V(X_s)|\, ds\right)\right] + c_0 c_1 M \\ &\leq 2 + c_0 c_1 M, \end{aligned} \tag{94}$$

since $|X(\tau_{B(y,a)})| \leq |x| + |x - y| + a \leq 2A + 2a \leq 4A$, the first inequality in (94) follows from (93).

Now for $x \in B^*_{2A}$ and $y \neq x$,

$$\begin{aligned} u_0(x, y) &= E^x_y[\xi \leq \tau_{B^*_{2A}}; e_V(\xi)] + E^x_y[\tau_{B^*_{2A}} < \xi; u_0(X(\tau_{B^*_{2A}}), y)] \\ &\leq E^x_y\left[\exp\left(\int_0^{\tau_{B^*_A}} |V(X_s)|\, ds\right)\right] + (2 + c_0 c_1 M) \\ &\leq 4 + c_0 c_1 M. \end{aligned} \tag{95}$$

Thus (f) follows from (93)–(95).

(f) ⇒ (c): Let $W \equiv \sup_{x, y \in R^d, x \neq y} u_0(x, y) < \infty$. then we have

$$(96)\quad \begin{aligned} u_0(x, y) - 1 &= E_y^x \left[\exp\left(- \int_0^\xi V(X_s)\, ds \right) - 1 \right] \\ &= -E_y^x \left[\int_0^\xi V(X_t) \exp\left(- \int_t^\xi V(X_s)\, ds \right) dt \right] \\ &= -E_y^x \left\{ \int_0^\xi V(X_t) E_y^{X_t}[e_V(\xi)]\, dt \right\} \\ &= -E_y^x \left[\int_0^\xi V(X_t) u_0(X_t, y) \right] \\ &= -\int_{R^d} \frac{G(x, z)V(z)G(z, y)}{G(x, y)} u_0(z, y)\, dz . \end{aligned}$$

The equalities in (96) can be justified by Fubini's Theorem and the inequalities

$$\begin{aligned} &\int_{R^d} \frac{G(x, z)|V(z)|G(z, y)}{G(x, y)} u_0(z, y)\, dz \\ &\leq W 2^{d-2} \left(c_d \int_{R^d} \frac{|V(z)|}{|x - z|^{d-2}}\, dz + c_d \int_{R^d} \frac{|V(z)|}{|y - z|^{d-2}}\, dz \right) \\ &\leq W c_d 2^{d-1} \, ||| \, V \, ||| < \infty . \end{aligned}$$

Put $F(x, y) = u_0(x, y) G(x, y) > 0$, $x \neq y$. Then by (96),

$$(97)\quad F(x, y) = G(x, y) - \int_{R^d} G(x, z) V(z) F(z, y)\, dz .$$

For $\psi \in C_c^\infty(R^d)$, put

$$F\psi(\cdot) \equiv \int_{R^d} F(\cdot, y)\psi(y)\, dy, \quad G\psi(\cdot) \equiv \int_{R^d} G(\cdot, y)\psi(y)\, dy .$$

By (97), $F\psi = G\psi - G[V(F\psi)] = G(\psi - V(F\psi))$. So we have $\Delta F\psi = \Delta G[\psi - V(F\psi)] = -2[\psi - V(F\psi)]$, i.e., $(-\Delta/2 + V)(F\psi) = \psi$. This means that $F(\cdot, \cdot)$ is the positive Green function $G^V(\cdot, \cdot)$ for H. Hence (c) holds.

(c) ⇒ (f1): For any ball $B = B(0, R)$, $R > 0$, we have

$$(98)\quad G_B^V(x, y) \leq G^V(x, y) < \infty, \qquad x, y \in B .$$

Then by Lemma 20 and (98), we have for any $x, y \in R^d$ with $x \neq y$ and $R > \max(|x|, |y|)$,

$$E_y^x[\xi < \tau_{B(0,R)}; e_V(\xi)] \leq \frac{G_B^V(x, y)}{G(x, y)} \leq \frac{G^V(x, y)}{G(x, y)} .$$

Letting $R \uparrow \infty$, By Fatou's lemma, we obtain $E_y^x[e_V(\xi)] \leq G^V(x, y)/G(x, y) < \infty$.

(f1) ⇒ (e): Suppose for some x_0, y_0 with $x_0 \neq y_0$, $u_0(x_0, y_0) < \infty$. By enlarging A and reducing a (if necessary), we may assume that $x_0 \in \overline{B_{4A}}$, $y_0 \in \overline{B_{8A}}$, and $0 < a \leq |x_0 - y_0|$. Thus by Lemma 18 and Lemma 19, we have $u_0(x_0) < \infty$, i.e., (e) holds. Thus we proved the equivalence among (e), (f), and (c).

Since (a) $\Rightarrow$ (b), obviously, and we have proved (g) $\Rightarrow$ (a), our last implication is

(b) $\Rightarrow$ (g): For each $t \in [0, 1+\beta]$, put

$$f(t) \equiv \ln[\lambda(tV)] = \lim_{n\to\infty} \frac{1}{n} \ln \|(S_{tV})^n\|. \tag{99}$$

Define a sequence of stopping times $\{T_k\}$:

$$T_1 = \tau_{B_{2A}} + \tau_{B_A^*} \circ \theta_{\tau_{B_{2A}}},$$

and

$$T_{k+1} = \begin{cases} T_k + T_1 \circ \theta_{T_k} & \text{if } T_k < \infty, \\ \infty & \text{if } T_k = \infty. \end{cases}$$

Then for each $n \geq 1$,

$$(S_{tV})^n f(x) = E^x[T_n < \infty; e_{tV}(T_n) f(X(T_n))],$$

and

$$\|(S_{tV})^n\| = \|(S_{tV})^n 1\| = \sup_{x\in\partial B_A} E^x[T_n < \infty; e_{tV}(T_n)].$$

Hence by the Cauchy-Schwarz inequality for any $t, s \in [0, 1+\beta]$, we have

$$\|S^n_{((t+s)/2)V}\| \leq \|S^n_{tV}\|^{1/2} \|S^n_{sV}\|^{1/2}.$$

Therefore the function $t \to \ln \|S^n_{tV}\|$ is convex, so is $f(t)$ by (99).

By Lemma 13, $\lambda(tV) \leq 1$, $t \in [0, 1+\beta]$, so $f(t) \leq 0$ on $[0, 1+\beta]$. Since for $V \equiv 0$, $E^x[e_0(\infty)] \equiv 1$, by Lemma 10, $\lambda(0) < 1$, i.e., $f(0) < 0$. We obtain that $f(1) < 0$, since otherwise the convex function f reaches its maximum at an interior point 1 in $[0, 1+\beta]$, f must be identical to 0 on $[0, 1+\beta]$, which contradicts $f(0) < 0$. Thus we obtain that $\lambda(V) = \exp[f(1)] < 1$. □

Postscript

The conjecture made in the Introduction was recently solved by V. Pinchover with an affirmative answer (a private communication).

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